

Continuous fuzzy measurement of energy for a two-level system

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Abstract

A continuous measurement of energy which is sharp (perfect) leads to the quantum Zeno effect (freezing of the state). Only if the quantum measurement is fuzzy, continuous monitoring gives a readout $E(t)$ from which information about the dynamical development of the state vector of the system may be obtained in certain cases. This is studied in detail. Fuzziness is thereby introduced with the help of restricted path integrals equivalent to non-Hermitian Hamiltonians. For an otherwise undisturbed multilevel system it is shown that this measurement represents a model of decoherence. If it lasts long enough, the measurement readout discriminates between the energy levels and the von Neumann state reduction is obtained. For a two-level system under resonance influence (which undergoes in absence of measurement Rabi oscillations between the levels) different regimes of measurement are specified depending on its duration and fuzziness: 1) the Zeno regime where the measurement results in a freezing of the transitions between the levels and 2) the Rabi regime when the transitions maintain. It is shown that in the Rabi regime at the border to the Zeno regime a correlation exists between the time dependent measurement

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readout and the modified Rabi oscillations of the state of the measured system. Possible realizations of continuous fuzzy measurements of energy are sketched.

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1 Introduction

Rabi flopping is a well known phenomenon: If a two-level atom with energy levels E_1 and E_2 is under the action of a monochromatic electromagnetic field, which we assume for simplicity to be resonant, then the state vector of the atom oscillates between the lower and upper state with the Rabi frequency which depends on the amplitude of the driving field. Speaking more precisely, the probability for the atom to be, say, on the level E_1 oscillates sinusoidally between 1 and 0. Whenever a von Neumann (“instantaneous” and perfect) measurement of energy is made at a certain time, the measurement result is either E_1 or E_2 with the probability depending on the time when the measurement is performed, in accordance with this sinusoid. The atom is after the measurement in the energy eigenstate corresponding to E_1 or E_2 . This means that such a measurement disturbs the Rabi oscillation radically: the phase of the Rabi flopping during next period is determined by the result of the von Neumann measurement.

It is also well known that if perfect measurements of energy of the atom (von Neumann measurements) are very frequent and in the limit continuous, the interaction with the measuring apparatus prevents atomic transitions between levels. The atom stays in an eigenstate and the readout of the continuous perfect measurement reduces to the constant value E_1 or E_2 . This is the often discussed quantum Zeno effect [1] which has been verified experimentally [2]. It is impossible in this regime of measurement to obtain information about the Rabi oscillation. Moreover, the Rabi oscillation is prevented by the measurement because the influence of the measurement on the atom is stronger than the influence of the driving field.

In contrast to this we want to address in the following the fundamental question if it is nevertheless possible to make the Rabi oscillation “visible” by *continuous* measurement of the atomic energy, i.e. to obtain a correlation between the measurement readout $E(t)$ and the Rabi oscillation of the state vector. To achieve our goal we must first of all avoid the Zeno effect by

weakening the influence of the measurement. This can be done with the help of continuous measurements which are fuzzy (unsharp) instead of being perfect (sharp).

A *fuzzy* (unsharp) measurement is caused by the low resolution of the respective detector. In the case of an observable with a discrete spectrum fuzziness means that the measurement is not able to transfer the state of the system to an eigenvector. Fuzziness does not go back to a defect detector, but to its finite resolution. Note that in practice a fuzzy measurement is the generic case and that a sharp or perfect measurement may be considered only as an idealization. The study of continuous fuzzy measurements opens a very interesting new field in which fundamental problems of quantum theory as well as potentially important concrete applications are discussed. There are physical situations in which we have to rely on fuzzy measurements if we want to obtain relevant information [3].

The theory of continuous quantum measurements has a long history. An incomplete list of references is [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The majority of authors studying continuous quantum measurements base their investigations on particular micro-models for the measuring device, see [8, 9, 10, 3] for the measurement of the energy of a two-level system. As compared to this the phenomenological and therefore model-independent approach presented below has the big advantage that the universal structures of the theory of continuous measurements can be revealed, including those connected with fuzziness. This approach based on restricted path integrals (RPI). It has been proposed in [6, 15]. The approach has already been applied to the measurement of energy in [16, 17]. However, an *ad hoc* assumption was made in [16, 17] that the measurement readout is a constant curve $E(t) = E_n = \text{const.}$ This is too restrictive and misleading for a continuous fuzzy measurement.

A theory of continuous fuzzy measurements (say, monitoring some observable has to answer the following questions: How can the type and strength of fuzziness be represented theoretically and what will be the characteristic parameters? Let the measurement of the observable $A(p, q, t)$ be of duration T , what is the probability to obtain the measurement readout $[a] = \{a(t) | 0 \leq t \leq T\}$? If for an initial state $|\psi_0\rangle$ of a system a particular measurement readout $[a]$ is obtained, what will be the state $|\psi_T^{[a]}\rangle$ of the system at the end of the measurement? The way to answer these questions in the framework of the phenomenological RPI approach will be presented

in Sect. 2. The resulting dynamics is governed by a complex Hamiltonian depending on $[a]$. The probability for $[a]$ can be read off from $|\psi_T^{[a]}\rangle$. We obtain as a characteristic property of a continuous fuzzy measurement that its resolution improves with increasing duration T as $\Delta a_T \sim 1/\sqrt{T}$. This reflects a Zeno-type influence which is still present even if it does not result in the proper Zeno effect.¹

In Sect. 3 the scheme is applied to the measurement of energy of a multilevel system which is not influenced otherwise (no additional driving field). Different durations T and the corresponding regimes of measurement are discussed. The time after which neighboring levels are resolved is called the level resolution time T_{lr} . Its shortness is a measure for the strength of a Zeno-type influence (“level freezing”) present in the continuous fuzzy measurement. The relation to the process of decoherence leading to the perfect (von Neumann) energy measurement is studied.

A two-level system under the resonant influence of a periodical driving field is then investigated in detail in Sect. 4. The driving field acts in opposition to the Zeno-type influence of the measurement lasting for the time T . Short Rabi periods $T_{\text{R}} \sim (\text{Rabi frequency})^{-1}$ represent strong influences of the driving field. It is obvious that different relative magnitudes of the times T_{lr} , T_{R} and T characterize different regimes of measurement with different behavior of the atom state. A complex general view is to be expected. We show that there is a regime in which a correlation exists between the time dependent measurement readout $E(t)$ and the Rabi oscillations of the atom state under the influence of the periodic driving field.

In Sect. 5 possible schemes of realization of the measurement in question are discussed. A continuous fuzzy measurement may be realized, according to these schemes, as a series of “instantaneous” (short-time) unsharp measurements. Finally in Sect. 6 concluding remarks are made.

2 A phenomenological approach to continuous fuzzy measurements

¹This phenomenon is referred to as “narrowing” by Milburn in the second of Ref. [9].

2.1 Restricted path integrals

Evolution of a quantum system when no measurement is performed (and therefore the system is closed) can be described by a unitary evolution operator U_t . The matrix element $\langle q''|U_t|q'\rangle = U_t(q'', q')$ of the evolution operator between states with definite coordinates (called the propagator) may be constructed with the help of the Feynman path integral (for example in the phase-space representation, i.e. with integration over paths $[p, q]$ in the phase space).

The formalism of path integrals is physically transparent and can be generalized to different non-standard situations. One of them is the situation when the system undergoes a *continuous measurement* (and is therefore open).² The idea to apply *restricted* path integrals (RPI) in this case was suggested by R.Feynman [18] and technically elaborated by one of the present authors (see [6] for the original papers and [15] for modern formulations).

A continuous measurement performed in the time interval $[0, T]$ gives some information about which paths the system propagates along. The central idea of the RPI approach is that the original path integral must be changed in accordance with this information. In the simplest case the path integral has to be restricted to a subset of paths depending on the measurement readout. In a more general case the contributing paths should be weighted by a weight functional depending on this readout.

To be more concrete, let us consider as a typical continuous fuzzy measurement the monitoring of an observable $A(p, q, t)$ during the time interval $[0, T]$. In this case the registered *measurement readout* (output) is a curve

$$[a] = \{a(t) | 0 \leq t \leq T\}. \quad (1)$$

Then the path integral has the form

$$U_T^{[a]}(q'', q') = \int d[p]d[q] w_{[a]}[p, q] e^{\frac{i}{\hbar} \int_0^T (p\dot{q} - H(p, q, t))}. \quad (2)$$

The weight functional $w_{[a]}$ will be discussed below.

Evolution of the system undergoing continuous fuzzy measurement resulting in a certain readout $[a]$ may now be represented by the following

²Of course, the measurement must be precise enough so that quantum effects be essential; then it is called a quantum measurement.

formulas:

$$|\psi_T^{[a]}\rangle = U_T^{[a]}|\psi_0\rangle, \quad \rho_T^{[a]} = U_T^{[a]} \rho_0 \left(U_T^{[a]}\right)^\dagger. \quad (3)$$

These formulas contain *partial propagators* depending on $[a]$. The first of these formulas may be applied to the evolution of pure states while the second is valid both for pure and mixed states. Note however that both of them are applicable only in the case when the measurement readout is known, i.e. in the situation of a *selective measurement*. A pure state remains pure after the selective measurement.

The norm of the state resulting from the measurement is, according to these formulas, less than unity. Normalization may of course be performed, but it is much more convenient and transparent to work - as we will do - with non-normalized states. Their norms give according to [15] the *probability densities* $P[a] = \text{Tr} \rho_T^{[a]}$ of the corresponding measurement readouts $[a]$. The following formula (with an appropriate measure of integration):

$$\text{Prob}([a] \in \mathcal{A}) = \int_{\mathcal{A}} d[a] P[a] = \int_{\mathcal{A}} d[a] \text{Tr} \left(U_T^{[a]} \rho_0 \left(U_T^{[a]} \right)^\dagger \right). \quad (4)$$

gives the probability for a measurement to result in an readout from the set \mathcal{A} . The word ‘density’ refers to the space of all measurement readouts.

In the considerations above we discussed the situation when the measurement readout is known (selective description of the measurement). If it is only known that the continuous fuzzy measurement is performed but its concrete readout is unknown (for example in *a priori* calculations), we have the situation of a *non-selective measurement*. In this case the evolution is described by the second of the formulas (3) with summation over all possible measurement readouts:

$$\rho_T = \int d[a] \rho_T^{[a]} = \int d[a] U_T^{[a]} \rho_0 \left(U_T^{[a]} \right)^\dagger. \quad (5)$$

The density matrix ρ_T is normalized for an arbitrary initial (normalized) state ρ_0 if the *generalized unitarity condition*

$$\int d[a] \left(U_T^{[a]} \right)^\dagger U_T^{[a]} = \mathbf{1} \quad (6)$$

is valid.

In the most general case we have only a partial information about the measurement readout. If we know that $[a] \in \mathcal{A}$, the evolution of the system must be described by the formula

$$\rho_T^{\mathcal{A}} = \int_{\mathcal{A}} d[a] \rho_T^{[a]} = \int_{\mathcal{A}} d[a] U_T^{[a]} \rho_0 \left(U_T^{[a]} \right)^\dagger \quad (7)$$

intermediate between Eq. (3) and Eq. (5).

To specify the type of fuzziness we are dealing with, we have to specify the weight functional $w_{[a]}$ in (2). To do so, we introduce the concept of closeness in the space of curves $[a]$ characterizing readouts of the continuous fuzzy measurement. By being *close* we shall mean that the temporal mean squared deflection abbreviated according to

$$\langle (A - a)^2 \rangle_T = \frac{1}{T} \int_0^T \left(A(p(t), q(t), t) - a(t) \right)^2 dt \quad (8)$$

is small. According to the Feynman approach $A(p, q, t)$ is a *c*-number function corresponding to the quantum observable. (8) is an integral characteristic of the difference between two functions. This characteristic is large (and the curves are therefore not close) only if the deflection of one curve from another is big during a long time. The curves which have large deflections but only during short intervals are considered to be close.³

Based on this, the information supplied by the readout $[a]$ of a continuous fuzzy measurement can be typically presented by the *Gaussian weight functional*

$$w_{[a]}[p, q] = \exp \left[-\kappa \int_0^T \left(A(p(t), q(t), t) - a(t) \right)^2 dt \right] \quad (9)$$

which is exponentially small if $\langle (A - a)^2 \rangle_T$ is big. The inverse of the parameter κ may serve as a *measure of fuzziness*. Later on, it will be connected with the resolution of these devices. Eq. (9) characterizes a particular class of measuring devices.⁴

³We accept here all curves $[a]$ as possible measurement readouts. In fact only comparatively slowly varying curves $[a]$ must be considered if the inertial properties of a real measuring device are taken into account. Then large deflections for a short time are impossible.

⁴The case of a sharp continuous measurement (when a single path presents a measurement readout) is considered in the paper of Aharonov and Vardi [19].

We shall consider the case when the fuzziness of the measurement does not alter with time, so that κ is constant. In principle, the parameter κ could depend on time (then it must be included in the integrand of the time integral). In any case κ cannot depend on the duration T of the measurement. This follows from group-theoretical properties of continuous measurements [15, 20], but can also be derived from models of measurements [14]. An important consequence of this fact will be discussed below, namely that the resolution of the continuous measurement is improved according to $\Delta a_T \sim 1/\sqrt{T}$ when its duration increases.

2.2 Complex Hamiltonian

Using the weight functional (9) in Eq. (2), we find for the partial propagators describing the evolution corresponding to a certain measurement output $[a]$

$$U_T^{[a]}(q'', q') = \int d[p] d[q] \exp \left\{ \frac{i}{\hbar} \int_0^T (p\dot{q} - H_{[a]}(p, q, t)) dt \right\} \quad (10)$$

where $p = p(t)$ and $q = q(t)$. The restricted path integral (10) has the form of a non-restricted Feynman path integral but with an *effective Hamiltonian*

$$H_{[a]}(p, q, t) = H(p, q, t) - i\kappa\hbar (A(p, q, t) - a(t))^2 \quad (11)$$

containing an imaginary term.⁵

The possibility to describe the dynamics by an effective Hamiltonian is very important since it makes practical applications rather simple: one may forget about path integrals and reduce the problem to the solution of the equivalent Schrödinger equation with a complex Hamiltonian. It is important also from the theoretical point of view because the imaginary part of the Hamiltonian indicates a characteristic trait of the process of continuous measurement namely that information is dissipated.⁶

⁵The form of the imaginary term may be different for another weight functional $w_{[a]}$, i.e. for another class of measuring devices.

⁶Let us remark that restricted path integrals describing continuous fuzzy measurements and leading to non-Hermitian Hamiltonians must not be confused with path integration in the presence of infinite potential barriers. Potential barriers also do not permit paths to go out of some corridor, but the unitary character of the process is not violated, no dissipation occurs.

2.3 Measurement of energy

We turn to the measurement of energy. Let the Hamiltonian H of the system have the form $H = H_0 + V$ where H_0 is the Hamiltonian of the “free” multilevel system and V is a potential describing an external influence leading to transitions between levels if no measurement is applied. By measurement of the energy we shall mean the measurement of the observable $A = H_0$ leading to the readout $[E] = \{E(t') | 0 \leq t' \leq T\}$.

The weight factor (9) with $A = H_0$ and $[a] = [E]$ can be written in the form

$$w_{[E]}[p, q] = \exp \left[-\frac{\langle (H_0 - E)^2 \rangle_T}{\Delta E_T^2} \right] \quad (12)$$

where we made use of the notation (8) and introduced the parameter

$$\Delta E_T = \frac{1}{\sqrt{\kappa T}} \quad (13)$$

characterizing the fuzziness of the measurement in a more transparent way than κ .

The weight factor (12) guarantees that only those Feynman paths $[p, q]$ are taken into account, for which the time dependence of the observable H_0 is presented by a curve close to the curve $[E]$. The measure of the deflection $\langle (H_0 - E)^2 \rangle_T$ is, according to (12), less or of the order of ΔE_T^2 . This means that the *resolution* of the measuring device described by our formulas is equal to ΔE_T .⁷

It has been stated above that the parameter κ does not depend on the duration T of the measurement. Therefore, it follows from Eq. (13) that the resolution of the continuous fuzzy measurement is improved when the duration of measurement increases. This important property may be understood if one replaces the continuous fuzzy measurement by a sequence of instantaneous fuzzy measurements with Gaussian weight functionals. It is plausible that such repetition of fuzzy measurements improves the resulting sharpness.

The measurement of energy is now worked out as follows: Let the measured system be initially in a pure state $|\psi_0\rangle$. Then for a particular given

⁷In the preceding papers on RPI approach in continuous measurements an analogous parameter was usually called measurement error. However, use of this term may be misleading, and now we prefer a more definite term “resolution”.

readout $[E]$ its further evolution is obtained by solving the corresponding Schrödinger equation (comp. (11) with $A = H_0$ and $[a] = [E]$)

$$\frac{\partial}{\partial t}|\psi_t\rangle = \left(-\frac{i}{\hbar}H - \kappa(H_0 - E(t))^2\right)|\psi_t\rangle. \quad (14)$$

The norm of the final state vector $|\psi_T\rangle$ gives then, according to the general formula (4), the probability density for the measurement readout $[E]$:

$$P[E] = \langle\psi_T|\psi_T\rangle. \quad (15)$$

This together with the resulting state $|\psi_T\rangle$ is what can be known about the measurement.

Expanding the state $|\psi_t\rangle$ in the basis

$$|\varphi_n(t)\rangle = e^{-iE_n t/\hbar}|n\rangle \quad (16)$$

of time-dependent eigenstates of H_0 , we have the following system of equations for the coefficients of the expansion $|\psi_t\rangle = \sum C_n(t)|\varphi_n(t)\rangle$:

$$\dot{C}_n = -\kappa(E_n - E(t))^2 C_n - \frac{i}{\hbar} \sum_{n'} \langle\varphi_n|V|\varphi_{n'}\rangle C_{n'}. \quad (17)$$

The probability density of the measurement readouts is given by

$$P[E] = \sum_n |C_n(T)|^2. \quad (18)$$

3 “Free” multilevel system: model of decoherence

We shall consider first a “free” multilevel system with the Hamiltonian H_0 . “Free” means vanishing V so that there is no driving field and no transitions between levels occur. We shall see in this simple case that a continuous fuzzy measurement of energy may serve as a model for decoherence leading to the resolution of the energy levels.

Because of $V = 0$, Eq. (17) have a simple solution

$$C_n(T) = C_n(0) \exp \left[-\kappa \int_0^T dt (E_n - E(t))^2 \right] = C_n(0) \exp \left[-\frac{\langle (E_n - E)^2 \rangle_T}{\Delta E_T^2} \right] \quad (19)$$

where $\langle \rangle_T$ is a time-average defined in (8) and ΔE_T is defined by Eq. (13).

3.1 Probability distribution

A probabilistic analysis may be based on Eqs. (4) and (6). In our case the probability density

$$P[E] = \sum_n |C_n(T)|^2 = \sum_n |C_n(0)|^2 \exp \left[-2 \frac{\langle (E_n - E)^2 \rangle_T}{\Delta E_T^2} \right] \quad (20)$$

is exponentially small when the temporal mean squared deflection $\langle (E_n - E)^2 \rangle_T$ of the function $[E]$ from each level E_n is more than ΔE_T^2 . On the contrary, the probability density is close to the maximum if $\langle (E_n - E)^2 \rangle_T$ is much smaller than ΔE_T^2 for the level E_n corresponding to a maximum value of $|C_n(0)|^2$.

Let us now turn to the probability itself. The generalized unitarity has in the case of the energy measurement the form $\int d[E] U_{[E]}^\dagger U_{[E]} = \mathbf{1}$. Taking matrix elements of this equality in the energy representation, we have an equivalent condition

$$\int d[E] \exp \left[-2 \frac{\langle (E - E_n)^2 \rangle_T}{\Delta E_T^2} \right] = 1 \quad (\text{for all } n). \quad (21)$$

The path integral here is of Gaussian type and can be evaluated. It may be shown that the (generalized) unitarity condition is fulfilled if the functional measure $d[E]$ is chosen to be $d[E] = \prod_t \sqrt{2\kappa dt/\pi} dE(t)$.

Because of an exponential in the integrand, only paths close enough to the level E_n contribute in (21). Namely, the integral which is analogous but restricted to the subset of readouts

$$\mathcal{A}_n = \{[E] \mid \langle (E - E_n)^2 \rangle_T \lesssim \Delta E_T^2\}, \quad (22)$$

is close to unity:

$$\int_{\mathcal{A}_n} d[E] \exp \left[-2 \frac{\langle (E - E_n)^2 \rangle_T}{\Delta E_T^2} \right] \sim 1. \quad (23)$$

The probability that the measurement output $[E]$ belongs to an arbitrary set \mathcal{A} is according to the general formula (4) and with the help of (20)

$$\text{Prob}([E] \in \mathcal{A}) = \int_{\mathcal{A}} d[E] \text{Tr} \left(U_{[E]} \rho_0 U_{[E]}^\dagger \right) = \sum_n p_n(\mathcal{A}) |C_n(0)|^2 \quad (24)$$

with

$$p_n(\mathcal{A}) = \int_{\mathcal{A}} d[E] \exp \left[-2 \frac{\langle (E - E_n)^2 \rangle_T}{\Delta E_T^2} \right]. \quad (25)$$

Comparing this with Eq. (23), we see that $p_n = 1$ if $\mathcal{A}_n \subset \mathcal{A}$. If, on the contrary, the set \mathcal{A} does not intersect with \mathcal{A}_n (or their intersection is small compared with \mathcal{A}_n) then $p_n = 0$.

3.2 Different Regimes of Measurement

As has already been said, the measurement resolution ΔE_T decreases for increasing duration T of the measurement. To define different regimes of the measurement, this resolution should be compared with the characteristic difference $\Delta E \sim E_{n+1} - E_n$ between energy levels.⁸ It is more convenient however to go over to the corresponding time parameters. Let T_{lr} be the duration at which the measurement resolution ΔE_T achieves the value ΔE ,

$$T_{\text{lr}} = \frac{1}{\kappa \Delta E^2}. \quad (26)$$

Then ΔE_T is larger or smaller than ΔE depending on whether T is smaller or larger in comparison with T_{lr} .

From Eq. (19) we have for the coefficients C_n at time $t = T_{\text{lr}}$

$$C_n(T_{\text{lr}}) = C_n(0) \exp \left[-\frac{\langle (E_n - E)^2 \rangle_{T_{\text{lr}}}}{\Delta E^2} \right]. \quad (27)$$

Therefore, the coefficient $C_n(t)$ decreases essentially during the time T_{lr} if the deflection of $E(t)$ from E_n is of the order or greater than ΔE . This is why the parameter T_{lr} may be called *level resolution time*. Additional arguments for this will be given below.

Let us consider now different durations T and the corresponding regimes of measurement. In the case $T \gg T_{\text{lr}}$ we have $\Delta E_T \ll \Delta E$. Now the probability density (20) is not negligible only if for most of the measurement duration

$$|E - E_n| \lesssim \Delta E_T \ll \Delta E \quad (28)$$

⁸If the difference between levels substantially depends on the number n of the level, one can draw corresponding conclusions concerning the measurement in a certain energy band referring to the value ΔE which is typical for this band.

for some n . The corresponding set of readouts \mathcal{A}_n determined by Eq. (22) is in this case a very narrow band close to the constant curve $E(t) \equiv E_n$. The width of this band is much less than ΔE , so that no other level $E_{n'}$ with $n' \neq n$ lies inside \mathcal{A}_n . Taking the set of paths \mathcal{A} in (24) in such a way that it includes \mathcal{A}_n but does not intersect with any other $\mathcal{A}_{n'}$, $n' \neq n$, we have $p_n = 1$ and $p_{n'} = 0$ for $n' \neq n$. Therefore it follows from (24) that

$$\text{Prob}(\mathcal{A}) = |C_n(0)|^2. \quad (29)$$

Thus, in the case $T \gg T_{\text{lr}}$ the measurement readout $[E]$ will be an almost constant curve close to one of the levels E_n , and the probability of it being close to a specified E_n is $|C_n(0)|^2$. We see that the continuous fuzzy measurement, if it lasts longer than T_{lr} , distinguishes (resolves) between the levels. This justifies once more the name *level resolution time* for the parameter T_{lr} . To find the state of the system after the measurement in the considered case ($T \gg T_{\text{lr}}$), we have to use Eq. (19) with the function $E(t)$ satisfying the restriction (28). We obtain

$$C_n(T) = C_n(0), \quad C_{n'}(T) = 0, \quad n' \neq n. \quad (30)$$

Thus, after a continuous fuzzy measurement of duration longer than T_{lr} the situation is similar to the one in the conventional description of the energy measurement as it is given in the von Neumann postulate: 1) The only possible measurement readouts $[E]$ are those close to one of the energy levels E_n , 2) the probability of $[E]$ being close to a particular E_n is $|C_n(0)|^2$ and 3) in the case of $[E]$ being close to E_n the system turns out to be in the energy eigenstate $|\varphi_n\rangle$ after the measurement.

The physical process leading to the determination of an energy eigenvalue (or of the eigenvalue of any other observable with discrete spectrum) has been investigated in the framework of different models and was called the process of decoherence [12, 21]. We see now that the continuous fuzzy measurement of energy fixes the energy level during T_{lr} . Therefore it presents a phenomenological description of the decoherence process leading to the determination of the specific energy eigenvalue. The characteristic mark of decoherence is the dying out the off-diagonal matrix elements of the density matrix. It is easily seen that it is this what really happens at the time T_{lr} because due to (27) all products $C_i C_j^*$ with $i \neq j$ become exponentially small for $T \gg T_{\text{lr}}$. The time period during which non-diagonal elements die out, is

usually called the decoherence time. We call it level resolution time because we consider other regimes of measurement too, where the parameter T_{lr} plays another role.

In this context the measurement performed during the time $T \ll T_{\text{lr}}$ might be considered as not yet finished. However this point of view is applicable only if the aim of the measurement is to distinguish between different levels. Continuous fuzzy measurement may however have as a goal to estimate the energy of the system with a resolution worse than ΔE . Then a period of measurement shorter than T_{lr} makes sense too. Let us turn to this regime.

We consider the case $T \ll T_{\text{lr}}$. This is equivalent to $\Delta E_T \gg \Delta E$, so that the set of paths \mathcal{A}_n determined by Eq. (22) includes functions $[E]$ in a very wide energy band around the constant curve $E \equiv E_n$. The width of this band is ΔE_T , so that it includes many other energy levels. Vice versa, if we define the set of readouts \mathcal{A} to contain all functions $[E]$ in the energy band of the width ΔE_T between E_{min} and E_{max} , then many energy levels E_n , $n_1 \leq n \leq n_2$ lie inside this band. For each of these levels intersection between \mathcal{A}_n and \mathcal{A} makes a significant part of \mathcal{A}_n and therefore $p_n(\mathcal{A}) \sim 1$ for $n_1 \leq n \leq n_2$. On the other hand, for n outside the interval $[n_1, n_2]$ intersection between \mathcal{A}_n and \mathcal{A} is small so that $p_n(\mathcal{A})$ is small. Making use of Eq. (24), we have approximately

$$\text{Prob}(\mathcal{A}) = \sum_{n=n_1}^{n_2} |C_n(0)|^2. \quad (31)$$

This regime of measurement does not distinguish between the energy levels so that the character of the measurement in this regime is essentially the same as in the case of a continuous spectrum.

4 The driven system: correlation between measurement readout and Rabi flopping

We consider now a system which is exposed to an external driving field. For simplicity let it be a 2-level system under resonance influence of a periodical force with frequency $\omega = \Delta E/\hbar$ where $\Delta E = E_2 - E_1$. The potential energy has then the following non-zero matrix elements for the states (16): $\langle \varphi_1 | V | \varphi_2 \rangle = \langle \varphi_2 | V | \varphi_1 \rangle^* = V_0$. Eq. (17) gives for the related expansion

coefficients

$$\begin{aligned}\dot{C}_1 &= -ivC_2 - \kappa(E_1 - E(t))^2 C_1, \\ \dot{C}_2 &= -ivC_1 - \kappa(E_2 - E(t))^2 C_2\end{aligned}\tag{32}$$

with $v = V_0/\hbar$.

If no measurement takes place, i.e. for $\kappa = 0$, such a system undergoes periodic transitions between the eigenstates (*Rabi oscillations* or *Rabi flopping*):

$$C_1(t) = R_1(t), \quad C_2(t) = R_2(t)\tag{33}$$

with

$$\begin{aligned}R_1(t) &= C_1(0) \cos vt - iC_2(0) \sin vt, \\ R_2(t) &= C_2(0) \cos vt - iC_1(0) \sin vt.\end{aligned}\tag{34}$$

The parameter $v = V_0/\hbar$ is equal to a half of the Rabi frequency and the corresponding time scale is the *Rabi period* $T_R = \pi/v = \hbar\pi/V_0$. In the time T_R the system inverts completely from one level to the other and returns to the original level.

Now the energy measurement is characterized by three different time scales: the level resolution time T_{lr} , the Rabi period T_R , and the duration of measurement T . With regard to physical interpretation one may look at them as follows: Small T_{lr} represents quick level resolution because of small fuzziness and therefore strong Zeno-type influence of the measurement. Small T_R corresponds to a strong influence of the driving field. For large T fuzziness effectively decreases. Let us consider different limiting cases for the relations between these times which will correspond to characteristic regimes of measurement.

If $T \ll T_R$, then the measurement duration is too short for the effect of the resonance influence to be observed. In this case we have in fact a “free” system as considered in Sect. 3. The two possible relations, $T_{\text{lr}} \ll T$ and $T \ll T_{\text{lr}}$, between the remaining parameters correspond to the cases when the measurement distinguishes between the levels and when it is too short to distinguish between themes discussed above in Sect. 3.

We turn now to the opposite inequality, $T_R \lesssim T$. Now the influence of the driving field becomes stronger. But how big the resulting influence on the state of the system is, still depends on the amount of fuzziness of

the measurement. We will discuss in detail the regimes $T_{\text{lr}} \ll T_{\text{R}}$ (Zeno regime in which influence of the measurement is strong enough to damp out the Rabi oscillations), the opposite regime $T_{\text{R}} \ll T_{\text{lr}}$ (Rabi regime in which Rabi oscillations maintain but there is no correlation between them and the measurement readout $[E]$) and the regime $T_{\text{R}} < 2\pi T_{\text{lr}}$ but with T_{R} , $2\pi T_{\text{lr}}$ and T of the same order. The latter is according to our goal the most interesting one, because the energy measurement readout turns out to be correlated to the oscillations of the state vector (modified Rabi oscillations). Some remarks concerning the regime $T_{\text{R}} \ll T_{\text{lr}} \ll T$ will be made.

4.1 Zeno regime of measurement

The first regime of measurement corresponds to the relation $T_{\text{lr}} \ll T_{\text{R}} \ll T$. One of its specific features results from the inequality $T_{\text{lr}} \ll T$ equivalent to $\Delta E_T \ll \Delta E$ or $\kappa T \Delta E^2 \gg 1$. Because of this relation the damping terms in Eq. (32) act effectively when $[E]$ differs from the corresponding level E_1 or E_2 by a value of the order of ΔE or larger. This leads to a decay of the coefficients C_1 , C_2 . According to (18) the corresponding measurement readout $[E]$ has low probability density.

This decay of the coefficients may be prevented (and therefore the output may have large probability density) only if $[E]$ is close to one of the levels E_1 or E_2 . Recall our definition of being close. Intervals when $E(t)$ deflects from the level more than by ΔE may occur because of the fuzziness of the measurement, but their complete duration should be much less than T_{lr} and therefore much less than T .

To verify that the probability density is actually large for and $[E]$ close to E_n , we consider a solution of Eq. (32) for the case $E(t) \equiv E_1$.⁹ Then Eq. (32) reads:

$$\dot{C}_1 = -i \frac{\pi}{T_{\text{R}}} C_2, \quad \dot{C}_2 = -i \frac{\pi}{T_{\text{R}}} C_1 - \frac{1}{T_{\text{lr}}} C_2. \quad (35)$$

These equations are easily solved to give

$$C_1 = a e^{\lambda_1 t} + b e^{\lambda_2 t}, \quad C_2 = i \frac{T_{\text{R}}}{\pi} (\lambda_1 a e^{\lambda_1 t} + \lambda_2 b e^{\lambda_2 t}) \quad (36)$$

⁹We simplify the consideration in this way, but the results are correct for the function $[E]$ close, not identical to level E_1 .

where

$$\lambda_{1,2} = -\frac{1}{2T_{\text{lr}}} \pm \sqrt{\frac{1}{4T_{\text{lr}}^2} - \frac{\pi^2}{T_{\text{R}}^2}} \quad (37)$$

and the coefficients a, b are determined by the initial conditions. For $T_{\text{lr}} < T_{\text{R}}/2\pi$ we have exponentially decaying solutions. They can be completely described in a generic case, but we shall consider only the limiting regime $T_{\text{lr}} \ll T_{\text{R}}$ when the level resolution is much more rapid than the Rabi oscillation. We have in this case $\lambda_1 = -1/T_{\text{lr}}$, $\lambda_2 = -\nu^2\lambda_1$, where $\nu = \pi T_{\text{lr}}/T_{\text{R}}$ is a small parameter, and the coefficients take the form

$$C_1(T) = (C_1(0) - i\nu C_2(0)) e^{-\nu^2 \frac{T}{T_{\text{lr}}}}, \quad C_2(T) = -i\nu C_1(T). \quad (38)$$

Estimating the probability density by the formula (18), we find

$$P(E \equiv E_1) = (|C_1(0)|^2 + \nu^2 |C_2(0)|^2) e^{-2\nu^2 \frac{T}{T_{\text{lr}}}}. \quad (39)$$

The exponential factor here is close to unity for a very wide range of T , up to the values of the order of $T_{\text{R}}^2/T_{\text{lr}}$. Omitting this factor and neglecting the term containing ν^2 , we have

$$P[E \equiv E_1] = |C_1(0)|^2. \quad (40)$$

In the same approximation we have

$$C_1(T) = C_1(0), \quad C_2(T) = 0. \quad (41)$$

We calculated the probability density for the constant curve ($E \equiv E_1$) being the measurement readout. The same conclusion will be valid for the curves close but not identical to this constant curve. The set of such curves form a narrow band of the width ΔE_T . Because of $\Delta E_T \ll \Delta E$ this band does not include the second level E_2 . Just as in Sect. 3.2, the probability of the measurement output to belong to this band is equal to $|C_1(0)|^2$. It is clear that an analogous consideration may be applied for the measurement readout close to the energy level E_2 .

Thus, for $T_{\text{lr}} \ll T_{\text{R}} \ll T$ we have found a very simple picture: i) Only those measurement outputs $[E]$ have high probability which are close to one of the constant curves $E \equiv E_1$ and $E \equiv E_2$; ii) the probability of the output

to be close to E_1 or E_2 is given by the initial values of the decomposition coefficients $|C_1(0)|^2$ or $|C_2(0)|^2$ correspondingly; iii) in case of the output being close to E_1 or E_2 the final state is correspondingly the eigenstate $|\varphi_1\rangle$ or $|\varphi_2\rangle$; iv) Rabi oscillations are completely damped away. It is evident that this picture reflects the dominant influence of the quantum Zeno effect. It is therefore justified to call this regime the *Zeno regime*.

4.2 Rabi regime of measurement

We turn now to the case $T_R \lesssim T_{lr}$ when Rabi oscillations are not damped. The three variants of this regime will be discussed.

1. We consider first the case $T_R \ll T \ll T_{lr}$. What can be said about the measurement output $[E]$? Returning to Eq. (32), we have to apply now the inequality $\kappa T \Delta E^2 \ll 1$ equivalent to $T \ll T_{lr}$. It shows that the damping terms are essential only for very large deviation of $[E]$ from both E_1 and E_2 . If $[E]$ lies in the band of the width $\Delta E_T = \Delta E \sqrt{T_{lr}/T}$ around E_1, E_2 , no damping occurs. The solution of Eqs. (32) have then the form of the Rabi oscillations (33) and the probability density for each of these measurement outputs $[E]$ is equal (up to the order of magnitude) to unity:

$$\text{Prob}[E] = |R_1(T)|^2 + |R_2(T)|^2 = 1. \quad (42)$$

All the curves $[E]$ in the mentioned wide band may (approximately) be considered as equally probable. Probability density essentially decreases only for the curves which deviate from both levels E_1, E_2 more than by the amount ΔE_T (and therefore much more than by ΔE). The measurement in this regime does not respect the discrete character of the energy spectrum. The duration of the measurement is too short to resolve the levels. The energy is measured with the error ΔE_T much greater than ΔE .

Turning to the corresponding state of the system we note that with the damping terms being negligible in the present regime, Eqs. (32) have a very simple solution (33) describing Rabi oscillations. These oscillations are not prevented by the continuous measurement performed in this regime. They are sinusoidal with Rabi period T_R . We shall call therefore this regime of measurement the *strong Rabi regime*. The oscillations of the atom state are not reflected in this case in the readout $[E]$.

2. The important variant of the Rabi regime is the one which shows a correlation between oscillations of the state vector and the measurement

readout. We shall demonstrate that this takes place for $T_R < 2\pi T_{\text{lr}}$ but with T_R , $2\pi T_{\text{lr}}$ and T being of the same order. . The analysis of Eqs. (32) is more complicated and less reliable in this case, but the process may be investigated with the help of the numerical simulation.

Let us rewrite Eq. (32) in the form

$$\dot{C}_1 = -ivC_2 - \epsilon_1 C_1, \quad \dot{C}_2 = -ivC_1 - \epsilon_2 C_2 \quad (43)$$

with

$$\epsilon_i(t) = \kappa(E_i - E(t))^2 = \frac{1}{T_{\text{lr}}} \left(\frac{E - E_i}{\Delta E} \right)^2. \quad (44)$$

Introduce instead of C_n the new functions:

$$S_n(t) = e^{\mathcal{E}(t)} C_n(t) \quad (45)$$

with

$$\mathcal{E}(t) = \frac{1}{2} \int_0^t (\epsilon_1(t) + \epsilon_2(t)) dt. \quad (46)$$

Then these functions satisfy the equations

$$\dot{S}_1 + ivS_2 + \frac{1}{2}\sigma S_1 = 0, \quad \dot{S}_2 + ivS_1 - \frac{1}{2}\sigma S_2 = 0 \quad (47)$$

whereby $\sigma = \epsilon_1 - \epsilon_2$. Eqs. (47) imply the decoupled second-order equations:

$$\begin{aligned} \ddot{S}_1 + (v^2 + \frac{1}{2}\dot{\sigma} - \frac{1}{4}\sigma^2) S_1 &= 0, \\ \ddot{S}_2 + (v^2 - \frac{1}{2}\dot{\sigma} - \frac{1}{4}\sigma^2) S_2 &= 0. \end{aligned} \quad (48)$$

We shall assume (this will be justified *a posteriori* for the most probable measurement readout $[E_{\text{prob}}]$) that i) the deviation $|E(t) - \bar{E}|$ with $\bar{E} = \frac{1}{2}(E_1 + E_2)$ is not much larger than ΔE during most of the time, i.e. goes beyond these limits only for a short time or as a small deviation and ii) the spectrum of the function $E(t)$ does not contain frequencies much higher than v (a half of the Rabi frequency). Then

$$\frac{|\sigma|}{v} \lesssim \frac{T_R}{2\pi T_{\text{lr}}}, \quad \frac{|\dot{\sigma}|}{v^2} \lesssim \frac{T_R}{2\pi T_{\text{lr}}}. \quad (49)$$

Accordingly, to zeroth order in the ratio $T_R/2\pi T_{\text{lr}}$ we neglect the two last terms in the brackets of Eq. (48). The functions S_n are then approximated by harmonic functions of frequency ν , i.e. by the Rabi oscillations R_n of (34).

We have therefore as a zeroth approximation

$$C_1(t) = e^{-\mathcal{E}(t)} R_1(t), \quad C_2(t) = e^{-\mathcal{E}(t)} R_2(t). \quad (50)$$

This approximation reflects the general oscillatory character of the evolution of the state $|\psi_t\rangle$, but it is too rough to use it for calculating the probability and for estimating the most probable measurement readout $[E]$. Instead, we shall use another procedure for this.

Consider again the functions $C_n(t)$ at an arbitrary time moment $0 \leq t \leq T$ and form the function

$$P(t) = |C_1(t)|^2 + |C_2(t)|^2. \quad (51)$$

The quantity $P(t)$ depends on the values $\{E(t')|0 \leq t' \leq t\}$ of the function $[E]$ in the moments preceding t . This quantity can be interpreted as the probability density of the corresponding measurement readout if the measurement is finished at time t . With increasing time t the value $P(t)$ decreases.

Let us analyze how the function $P(t)$ depends on time and choose the function $[E]$ in such a way that $P(t)$ decreases as slowly as possible. Then the resulting function $[E]$ will correspond to the most probable measurement readout.

Differentiating Eq. (51) and making use of the Eq. (32) we have

$$\dot{P}(t) = -2\kappa \left[(E_1 - E(t))^2 |C_1|^2 + (E_2 - E(t))^2 |C_2|^2 \right]. \quad (52)$$

We minimize the absolute value of the r.h.s. and obtain for the most probable measurement output

$$E_{\text{prob}}(t) = \frac{E_1 |C_1(t)|^2 + E_2 |C_2(t)|^2}{|C_1(t)|^2 + |C_2(t)|^2}. \quad (53)$$

Now we can replace C_n by the approximate solution (50). This gives for the most probable measurement output

$$E_{\text{prob}}(t) = E_1 |R_1(t)|^2 + E_2 |R_2(t)|^2. \quad (54)$$

This curve is correlated with the Rabi oscillations of the state in the sense that the curve $E_{\text{prob}}(t)$ is closer to the level E_n when the probability for the system to be on this level is larger. The function $E_{\text{prob}}(t)$ oscillates around the middle line \bar{E} between the levels. The amplitude of the oscillations is equal to $\frac{1}{2}\Delta E\sqrt{(|C_1(0)|^2 - |C_2(0)|^2)^2 + 4(\text{Im}(C_1C_2^*))^2}$. If the initial value of the decomposition coefficient $C_1(0)$ is equal to $C_2(0)$, the most probable measurement output E_{prob} coincides with the constant curve $E \equiv \bar{E}$. Only in this case the Rabi oscillations disappear.

The measurement readout (54) is shown to be most probable. However, it is not clear from the above argument how rapidly the probability density decreases if the readout $[E]$ deviates from its most probable shape. If it decreases rapidly enough, then we may speak about correlation between the state oscillations and measurement readout. This is investigated with the help of the numerical calculation. We omit details. The following conclusions are obtained:

- i) Oscillations of the state vector between the energy eigenstates (Rabi regime) takes place for $T_R \lesssim 2\pi T_{\text{lr}}$. The boundary to the Zeno regime is marked by $T_R \approx 2\pi T_{\text{lr}}$.
- ii) The state oscillations are modified Rabi oscillations. Their frequency becomes smaller than the Rabi frequency when T_R approaches $2\pi T_{\text{lr}}$. Together with this the pure sinusoidal shape of the oscillations is lost. This indicates that there is already some Zeno-type influence of the measurement. See the figure for more details.
- iii) Correlation between the modified Rabi oscillations and the measurement readout $[E]$ is substantial for T_R close to $2\pi T_{\text{lr}}$. In the other case the band of different readouts $[E]$ having large probability becomes too wide to speak about a correlation.

3. The complex case $T_R \ll T_{\text{lr}} \ll T$ is not treated in detail here. We restrict ourselves to some comments. In this case small errors related to the approximate character of (50) grow exponentially, so that this solution is inappropriate for the analysis, and the conclusions made above are not valid. The evolution is qualitatively different in this case. Under these conditions the system “forgets” because of the measurement its initial conditions and the oscillating curve (54) cannot arise as a typical measurement output.

The mechanism of “forgetting” is loss of coherence leading to a mixture instead of a pure state. If the duration T of the measurement is not too long, the result of the measurement and the evolution of the system may be adequately presented by a single readout $[E]$ and the resulting pure final state $|\psi_T^{[E]}\rangle$. A single readout is a set of null measure and strictly speaking is characterized by zero probability. One has to consider a band of readouts instead and integrate the probability density $P[E]$ over this band to obtain non-zero probability. However close measurement readouts $[E]$ lead to close final states of the system if the measurement is not too long. Therefore, the final state corresponding to the band of readouts will be the same pure state $|\psi_T^{[E]}\rangle$ which corresponds to each of the readouts in the band.

This argument is not valid however for a long measurement. In this case even very close readouts $[E]$ lead to different final states $|\psi_T^{[E]}\rangle$. Therefore, even a very thin band of readouts gives a mixed final state. With increasing T this mixed state becomes universal, not depending on the initial state of the system.

Indirectly the latter statement is confirmed by the non-selective description of the measurement with the help of the density matrix (5). It has been shown in [22] that this density matrix satisfies a master equation having the form of the Liouville equation but with an additional term proportional to the double commutator $[A[A\rho]]$. In our case this equation reads as follows

$$\dot{\rho} = -\frac{i}{\hbar}[H_0 + V, \rho] - \frac{\kappa}{2}[H_0[H_0\rho]] \quad (55)$$

and can easily be solved for a resonantly driven two-level system. This gives for $8\pi T_{\text{lr}} > T_{\text{R}}$ (Rabi regime)

$$\rho_{11} - \rho_{22} = e^{-(t/4T_{\text{lr}})}(a \cos \Omega t + b \sin \Omega t). \quad (56)$$

Here matrix elements are taken between the states (16), $\Omega = \sqrt{(2v)^2 - (2T_{\text{lr}})^{-2}}$ and the parameters a, b are determined by initial conditions. It is evident that the difference $\rho_{11} - \rho_{22}$ due to non-symmetrical initial conditions disappears during time of the order of T_{lr} .

5 Possible realizations of continuous fuzzy measurements of energy

We want to sketch briefly possible realizations of continuous fuzzy measurements of energy. Our suggestions need more detailed elaboration which cannot be given here. An output of an instantaneous measurement is a number. An output of a sequential measurement is a series of numbers $a_1, a_2, \dots, a_n, \dots$. If the time scale is larger than the interval between instantaneous measurements this series of numbers may be identified with a continuous curve $a(t)$ presenting the readout of a continuous measurement (monitoring). Fuzziness of the continuous measurement depends on the fuzziness of the instantaneous measurements and on the intervals between them. We give three examples:

1. The most obvious approach is to make a continuous sharp measurement fuzzy in making the underlying instantaneous measurements fuzzy, in other words, to modify measurements leading to the quantum Zeno effect in this way. An arrangement for the Zeno effect was suggested in [23] and used in [2]. A system (ion in a trap) consisting of levels 1 and 2 is driven between these levels by a resonant perturbation. There is a subsidiary level 3 that can decay only to level 1. The state measurement is carried out by action of an optical pulse in resonance between levels 1 and 3. This pulse can induce the $1 \rightarrow 3$ transition with the subsequent spontaneous return to level 1 accompanied by the emission of a photon with the same frequency. It causes a projection of the ion onto level 1 accompanied by scattering the photon. This happens with the probability $|c_1|^2$ if the initial state of the ion is $c_1|1\rangle + c_2|2\rangle$. With the probability $|c_2|^2$ the pulse projects the ion onto level 2 with no scattering.

The measurement of this type is sharp (of von Neumann type) and answers the question whether the ion is on level 1. Frequent repetition of this measurement was shown [2] to lead to the quantum Zeno effect freezing the atom on level 1. Each measurement in a series completely resolves in this case between the energy levels, i.e. is not fuzzy. It is necessary to modify the experimental setup to realize a fuzzy measurement of energy.¹⁰

To do this, one may choose an ion with very fast decay of the subsidiary

¹⁰The repeated measurement of the same type but with varying parameters of the system has been considered in [10]. It is however difficult to compare the results obtained in this paper with our results because fuzziness was not the aim of investigation in [10].

level 3 to both main levels 1 and 2 and a continuous spectrum of the radiation inducing the transitions $1 \rightarrow 3$ and $2 \rightarrow 3$. The width of level 3 and the width of the spectrum of the inducing radiation have to be more than the difference ΔE between levels 1 and 2. The induced transition from levels 1 and 2 to the subsidiary level 3 with the subsequent spontaneous transition back to the same level will be accompanied by elastic scattering of a photon. Measuring the frequency of the scattered photon is equivalent in these conditions to an instantaneous fuzzy measurement of energy (performed with the resolution worse than ΔE). A series of scatterings is then a continuous fuzzy measurement of energy. Resolution of the continuous measurement ΔE_T depends on the length of the series or duration T of the continuous measurement. Special precaution should be taken in this scheme to provide that only transitions $1 \rightarrow 3 \rightarrow 1$ and $2 \rightarrow 3 \rightarrow 2$ might occur. For this, level 3 may be double, consisting of two very close levels $1'$ and $2'$ with the only transitions $1 \leftrightarrow 1'$ and $2 \leftrightarrow 2'$ permitted.

2. An electron in a magnetic field is a two-level system and repeated measurements of its energy can be performed with the help of several Stern-Gerlach (SG) devices. The position of the electron after scattering in the SG device depends on its energy. Therefore measuring the final position of the electron is in fact measuring its energy. If the initial position is uncertain to some degree, the resulting measurement of energy is fuzzy. In fixing the initial uncertainty, we can determine the fuzziness of the instantaneous measurement of energy. If now the electron is scattered subsequently in a series of SG devices, we have in fact a series of fuzzy instantaneous measurements of energy approximating a continuous one.

As a driving field we can include an external electromagnetic field in resonance with the difference between the electron levels. This induces periodic transitions between the levels. Then all the effects discussed in Sect. 4 can be observed with such a system. An analysis is given in [8, 3]. The results are in agreement with our general results.

3. One more way of introducing fuzziness can be found in quantum optics. The multilevel “free” system is in this case realized as a mode of the electromagnetic field. The measurement of energy of this system (usually referred to as the measurement of the photon number) is realized by the interaction of this mode with another (probe) electromagnetic wave through a nonlinear optical Kerr medium in which both waves propagate [24]. The

interaction Hamiltonian has the form

$$H_I = \hbar\chi (a^\dagger a) (b^\dagger b) \quad (57)$$

where a , b are the annihilation operators of the two modes. The observable to be measured indirectly is $H_0 = a^\dagger a$, but the directly measured observable is a quadrature phase component

$$b_2 = \frac{1}{2i}(b - b^\dagger). \quad (58)$$

After the probe wave has travelled the distance L in the Kerr medium, the value of this component is, due to the interaction H_I ,

$$b_2(L) \simeq \frac{\chi L}{c} \langle b_1(0) \rangle H_0 + \Delta b_2(0) \quad (59)$$

where $\Delta b_2(0)$ is an initial uncertainty of the amplitude b_2 and $\langle b_1(0) \rangle$ an initial mean value of the second quadrature component $b_1 = \frac{1}{2}(b + b^\dagger)$. This means that the output for the measurement of H_0 is read off as the value of the component b_2 . Fuzziness of the measurement depends therefore on $\Delta b_2(0)$ and can be controlled. The series of the measurements of this type may be considered as a continuous fuzzy measurement of energy H_0 .

6 Conclusions

Fuzziness of a measurement is caused by the low resolution of the detector. Such a measurement is unable to transfer the state of the system to an eigenstate of the measured observable, and the measurement readout must not agree with an eigenvalue. Such a continuous fuzzy measurement as monitoring an observable may be approximated by as a series of unsharp or imperfect “instantaneous” measurements. It is qualitatively different from a series of sharp (von Neumann type) measurements.

We introduced fuzziness with the help of restricted path integrals, on the basis of the resulting non-Hermitian Hamiltonians. We discussed the continuous fuzzy measurement (monitoring) of energy for a multilevel and in more detail for a two-level system leading to a readout $E(t)$. In addition to the duration T of the measurement there are two other characteristic time

scales. The level resolution time T_{lr} represents the strength of an inherent Zeno-type influence (“level freezing”). If an external resonant driving field is acting upon the system, the strength of its influence is represented by the Rabi period T_{R} . According to the relative orders of magnitudes of these times, different regimes with characteristic properties of the measurement readout and the related evolution of the state vector can be distinguished.

We obtained the following results:

- i) The study of a “free” multilevel system which is only under the influence of the measurement (no driving field present) shows that continuous fuzzy measurement of energy is a model for decoherence leading to the resolution of separate levels.
- ii) The discussion of a resonantly driven two-level system reveals two different regimes of measurement: In the Zeno regime the oscillations of the state vector are prevented and the measurement readouts are essentially discrete. In the Rabi regime the state oscillations maintain and the readouts may be arbitrary curves.
- iii) An important new effect is predicted: For time scales corresponding to the Rabi regime but being close to the border with the Zeno regime the frequency of the state oscillations is decreased and their shape deviates from pure Rabi oscillations. Correlation are found between the oscillations of the state vector and the measurement readouts. Thus, in following $E(t)$ one can directly read off the oscillations of the state vector.

Possible ways of realization of a fuzzy continuous measurement of energy are shortly commented.

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References

- [1] B. Misra and E. C. G. Sudarshan, J.Math.Phys. **18**, 756 (1977); C. B. Chiu, E. C. G. Sudarshan and B. Misra, Phys. Rev. **D 16**, 520 (1977); A. Peres, Amer. J. Phys. **48**, 931 (1980).
- [2] W. M. Itano, D. J. Heinzen, J. J. Bollinger and D.J. Wineland, Phys. Rev. **A 41**, 2295 (1990).
- [3] A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer Academic Publishers, Dordrecht, Boston & London, 1993.
- [4] H. D. Zeh, in: *On the irreversibility of time and observation in quantum theory*, Enrico Fermi School of Physics **II**, Academic Press, 1971, p.263.
- [5] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press: London, New York, San Francisco, 1976; M. D. Srinivas, *J.Math. Phys.* **18**, 2138 (1977).
- [6] M. B. Mensky, Phys. Rev. **D 20**, 384 (1979); Sov. Phys.-JETP **50**, 667 (1979).
- [7] A. Barchielli, L. Lanz and G. M. Prosperi, Nuovo Cim., **B 72**, 79 (1982).
- [8] A. Peres, Continuous monitoring of quantum systems, in *Information Complexity and Control in Quantum Physics*, ed. by A. Blaquiére, S. Diner, and G. Lochak, Springer Verlag, Wien, 1987, pp. 235-240.
- [9] D. F. Walls and G. J. Milburn, Phys. Rev. **A 31**, 2403 (1985); G. J. Milburn, J. Opt.Soc. Am. **B 5**, 1317 (1988).
- [10] M. J. Gagen and G. J. Milburn, Phys. Rev. **A 47**, 1467 (1993).
- [11] D. F. Walls, M.J. Collett, and G. J. Milburn, Phys. Rev. **D 32**, 3208 (1985); G. J. Milburn, Phys. Rev. **A 36**, 5271 (1987); H. M. Wiseman and G. J. Milburn, Phys. Rev. **A 47**, 642 (1993); M. J. Gagen, H. M. Wiseman, and G. J. Milburn, Phys. Rev. **A 48**, 132 (1993).
- [12] E. Joos, and H. D. Zeh, Z.Phys. **B 59**, 223 (1985).
- [13] L. Diosi, Phys.Lett. **A 129**, 419 (1988).

- [14] A. Konetchnyi, M. B. Mensky and V. Namiot, Phys. Lett. **A 177**, 283 (1993).
- [15] M. B. Mensky, *Continuous Quantum Measurements and Path Integrals*, IOP Publishing: Bristol and Philadelphia, 1993.
- [16] R. Onofrio, C. Presilla, and U. Tambini, Phys. Lett. **A 183**, 135 (1993).
- [17] U. Tambini, C. Presilla, R. Onofrio, Phys. Rev. **A 51**, 967 (1995).
- [18] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).
- [19] Y. Aharonov and M. Vardi, Phys. Rev. **D21**, 2235 (1980).
- [20] M. B. Mensky, Phys. Lett. **A 150**, 331 (1990).
- [21] W. H. Zurek, Phys. Rev. **D 24**, 1516 (1981); **D 26**, 1862 (1982).
- [22] M.B.Mensky, Phys. Lett. **A 196**, 159 (1994).
- [23] R.J. Cook, Phys. Scr. **T 21**, 49 (1988).
- [24] Y.Yamamoto, The Transactions of the IEICE, **E 73**, 1598 (1990).

Figure capture

Correlation between modified Rabi oscillations of the state of a resonantly driven two-level system and the readouts of a continuous fuzzy measurement of energy. We assume $T_R/2\pi T_{\text{lr}} = 0.8$. a) The in the upper diagram shows the periodic oscillations of the state vector between the two energy eigenstates. The period is close to but larger than the Rabi period T_R . The shape is of a slightly modified Rabi oscillation. b) The diagram in the middle shows as the dashed line the most probable energy readout $[E_{\text{prob}}]$ which is strongly correlated to the state oscillation. Two bands around it (with the width $W = 2$ and $W = 3$ are drawn. c) The probability P that an energy readout $[E]$ lies in the band with the width W around the most probable readout is given in the lower diagram.

